Announcements

1) HW 2 due Monday
   9/30

- MLC workers
- Interest in financial trader/analyst jobs?

- Inverse properties

- Return HW

The Hausdorff Maximality Principle

Given a set S and a partial ordering "$\leq$" on S, if T is any collection of subsets of S that is totally ordered, then $\exists$ a maximal totally ordered collection M containing T:

Maximal totally ordered set:

If $\mathcal{Y}$ is a collection of subsets of S and $X \in M$ $\Rightarrow X \in \mathcal{Y}$ then either $\mathcal{Y} = M$ or $\mathcal{Y}$ is not totally ordered.

Theorem: Let V be a vector space over $\mathbb{F}$. Then V has a basis.

proof: Take $x \in V$.

If $V =$ the zero vector space, then the single element 0_V serves.

Take $x \in V$, $x \neq 0_V$.

If $\forall\, y \in V$, $\exists\, q \in \mathbb{F}$

with $y = qx$, then

$\{x\}$ is a basis for

V. ($V = \mathbb{F}$, in some

sense).

Suppose $\exists$ at least two elements x and y that are linearly independent.

Define a partial order on the linearly independent subsets of V by set inclusion.

Take any totally ordered collection of linearly independent subsets of V.

Then by the Hausdorff Maximality principle, there is a maximal totally ordered collection M containing it.

Take the union of all elements in M. Call this union B.

Claim: B is a basis for V.

Linear Independence

Take $x_1, x_2, \dots, x_n \in B$

Since M is totally ordered, $\exists$ a linearly independent subset R in M containing $x_1, x_2, \ldots, x_n$ (the proof is by induction on n). Then $\{x_1, \ldots, x_n\} \subseteq R$, so $\{x_1, \ldots, x_n\}$ is linearly independent.

Spanning We've shown that $\mathcal{B}$ is linearly independent. We must have $\mathcal{B} \in M$, otherwise by adjoining $\mathcal{B}$, we have a totally ordered superset of M, since if $A \in M$, $A \subseteq \mathcal{B}$, so we preserve the total order.

Consider $x \in V$, $B \cup \{x\}$.

We claim this is not a basis for V if $x \notin B$.

$B \cup \{x\}$ contains every element of M as a subset.

Then by taking M' to be the collection of all sets in M along with $B \cup \{x\}$, we obtain a totally ordered collection of subsets of V that contains M, contradicting the maximality of M.

Therefore $\operatorname{span}(B) = V$, and so B is a basis. $\square$

Note: in more restrictive situations, we can get away with not using the Hausdorff Maximality Principle.

Facts about the Maximality Principle

The axioms that govern "naive set theory" are the Zermelo-Frankel axioms (ZF). The maximality principle cannot be deduced from them - nor can its negation!

Statements logically equivalent to the Maximality Principle:

1) The well-ordering theorem

2) Axiom of Choice

3) Zorn's Lemma

4) Existence of bases in vector spaces (!!)